Two-Sided Alternative

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Introduction

- Let $X \sim N(\mu, \sigma^2)$ where σ^2 is known, Consider the hypothesis

$$H_0 : \mu = \mu_0 \;\; against \;\; H_1 : \mu \neq \mu_0$$

- Here the alternative implies that $\mu < \mu_0 \;\; or \;\; \mu > \mu_0$. Therefore it is a two sided alternative.
- A possible test procedure for $H_0 : \mu = \mu_0$ against the two sided alternative is based on the critical region.
- $$\omega = \{\bar{x} < c_1 \;\; or \;\; \bar{x} > c_2 | H_0\}$$

  where $c_1 \;\; c_2$ are constants to be determined such that

$$P[X \in \omega | H_0] = \alpha$$

Cont'd

- This means that c_1 and c_2 satisfy

$$P[\bar{X} < c_1 \;\; or \;\; \bar{X} > c_2 | H_0] = \alpha$$

- $$P[\bar{X} < c_1 | H_0] + P[\bar{X} > c_2 | H_0] = \alpha$$

- $$P\left[\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} < \frac{c_1 - \mu_0}{\sigma/\sqrt{n}}\right] + P\left[\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} > \frac{c_2 - \mu_0}{\sigma/\sqrt{n}}\right] = \alpha$$

- $$P\left[z < \frac{c_1 - \mu_0}{\sigma/\sqrt{n}}\right] + P\left[z > \frac{c_2 - \mu_0}{\sigma/\sqrt{n}}\right] = \alpha$$

Cont'd

- One way of choosing c_1 and c_2 is such that:

$$P\left[z < \frac{\sqrt{n}(c_1 - \mu_0)}{\sigma}\right] = P\left[z > \frac{\sqrt{n}(c_2 - \mu_0)}{\sigma}\right] = \frac{\alpha}{2}$$

  This is a case of two sided equal tailed test.

- Then

$$\frac{\sqrt{n}}{\sigma}(c_1 - \mu_0) = -z_{\frac{\alpha}{2}} \rightarrow c_1 = \mu_0 - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

  and

$$\frac{\sqrt{n}}{\sigma}(c_2 - \mu_0) = -z_{\frac{\alpha}{2}} \rightarrow c_2 = \mu_0 + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

Cont'd

- Thus the two sided equal tailed test is to reject $\mu = \mu_0$ whenever

$$\bar{x} < \mu_0 - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \quad or \quad \bar{x} > \mu_0 + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

- We note that this two sided hypothesis has two different critical regions:

$$\omega_1 = \{x : \bar{x} < \mu_0 - \frac{\sigma}{\sqrt{n}} Z_{\frac{\alpha}{2}}\}$$

and

$$\omega_2 = \{x : \bar{x} > \mu_0 + \frac{\sigma}{\sqrt{n}} Z_{\frac{\alpha}{2}}\}$$

- Since the two regions are different there exists no critical region of size α which is UMP for testing the two sided alternative.

Example

Suppose that $X_1, X_2, ..., X_n$ from a random sample from a normal distribution for which μ is unknown and the variance $\sigma^2 = 1$. Suppose also that μ_0 is a certain specified number and that the following hypothesis are to be tested:

$$H_0 : \mu = \mu_0 \;\; against \;\; H_1 : \mu \neq \mu_0$$

Finally suppose that the sample size $n = 25$. Consider a test procedure such that H_0 is to be accepted if $|\bar{X} > \mu_0| < c$. Determine the value of c such that the size of the test will be 0.05

Solution

- $X \sim N(\mu, 1), n = 25, \bar{x} \sim N(\mu, \frac{1}{n})$
- $$\begin{aligned} \bar{\omega} = \{X : |\bar{x} - \mu_0| < c\} = \alpha = 0.05 \\ P[X \in \omega | H_0] &= 0.05 \\ &= 1 - P[X \in \bar{\omega} | H_0] = 0.05 \\ &= P(X \in \bar{\omega} | H_0) = 0.95 \\ &= P[|\bar{X} - \mu| < c | H_0] = 0.95 \qquad (1) \\ &= P[-c < \bar{X} - \mu < c | H_0] = 0.95 \\ &= P\left[\frac{-c}{1/\sqrt{n}} < \frac{\bar{x} - \mu_0}{1/\sqrt{n}} < \frac{c}{\frac{1}{\sqrt{n}}}\right] = 0.95 \end{aligned}$$

Solution Cont'd

- $$P[-\sqrt{n}c < z < \sqrt{n}c] = 0.95$$

- $$P[-5c < z < 5c] = 0.95$$

- $$5c = z_{\frac{\alpha}{2}} \to c = \frac{z_{\frac{\alpha}{2}}}{5} = \frac{1.96}{5} = 0.392$$

Thank You!